CS-466/566: Math for AI

Module 01: MinMax Optimization

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The Tax Revenue Problem

Imagine you're elected prime minister. You want to **maximize tax revenues**.

- If tax rate = **0%** → Revenue = **$0** (nothing collected)
- If tax rate = **100%** → Revenue ≈ **$0** (nobody works)

The Sweet Spot: Somewhere between 0% and 100% lies the rate that **maximizes** revenue.

But where exactly?

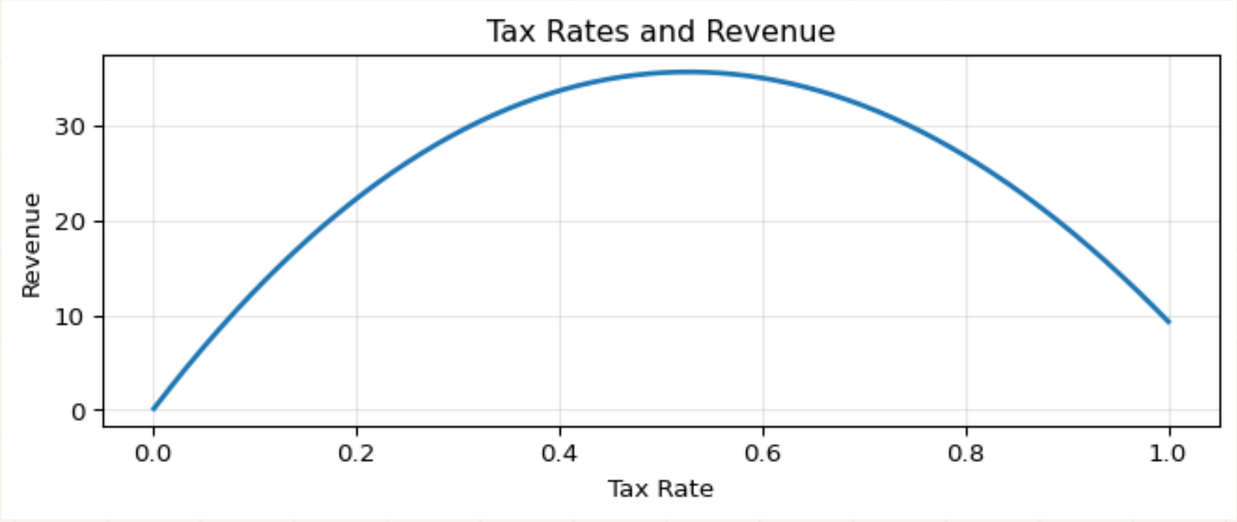
Goal: Find x^* such that $f(x^*) \geq f(x)$ for all x.

The Revenue Function

Your economists derive a revenue model:

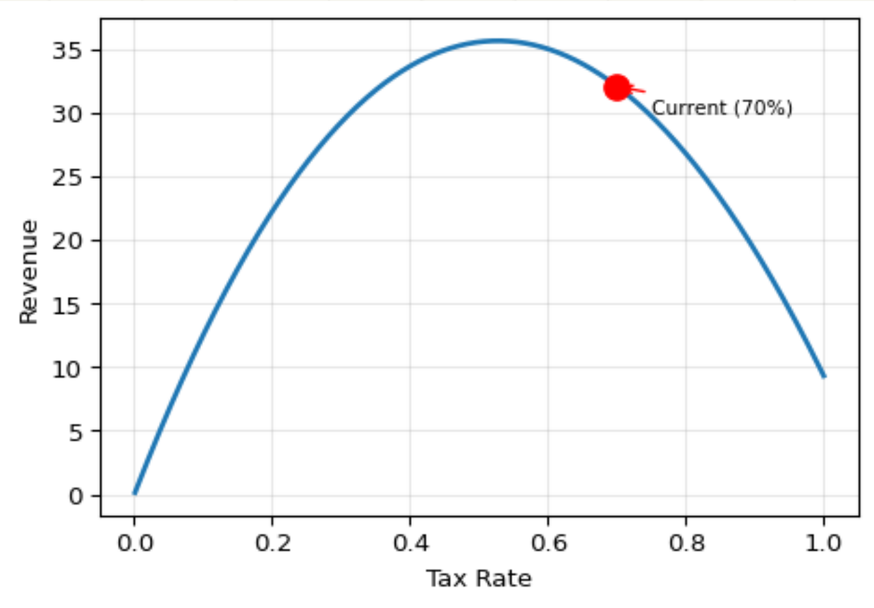
```python
import math
def revenue(tax):
    return 100 * (math.log(tax + 1) - (tax - 0.2)**2 + 0.04)
```

Do not worry about the exact form of the function, just know that it is a function that we want to maximize. It takes a single input, the tax rate, and returns the revenue.



Where Are We Now?

Your country currently has a **70% flat tax**. Is this optimal?




```python
current_rate = 0.7

xs = np.linspace(0.001, 1, 500)
ys = [revenue(x) for x in xs]

fig, ax = plt.subplots(figsize=(5, 3.5))
ax.plot(xs, ys, linewidth=2)
ax.plot(current_rate,
        revenue(current_rate),
        'ro', markersize=10)
ax.annotate('Current (70%)',
    xy=(current_rate,
        revenue(current_rate)),
    xytext=(0.75, 30), fontsize=8,
    arrowprops=dict(arrowstyle='->',
                    color='red'))
ax.set_xlabel('Tax Rate')
ax.set_ylabel('Revenue')
ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()
```

We need a **precise** answer — not just a guess from the plot!

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What is a Derivative?

The derivative of f at a point x measures the **slope of the tangent line**:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Interpretation:

- $f'(x) > 0$: function is **increasing** → move right to go up
- $f'(x) < 0$: function is **decreasing** → move left to go up
- $f'(x) = 0$: **critical point** → possible maximum or minimum!

Numerical Approximation

When we can't compute $f'(x)$ analytically, we **approximate** it:

Forward difference: $\frac{f(x+h)-f(x)}{h}$ — error is $O(h)$

Central difference: $\frac{f(x+h)-f(x-h)}{2h}$ — error is $O(h^2)$ ✅

Central difference is **more accurate**: it cancels first-order error terms.

Derivative of the Revenue Function

Our revenue function: $f(x) = 100\left[\ln(x+1) - (x-0.2)^2 + 0.04\right]$

Applying calculus rules:

$$f'(x) = 100\left[\frac{1}{x+1} - 2(x-0.2)\right]$$

```python
def revenue_derivative(tax):
    return 100 * (1/(tax + 1) - 2 * (tax - 0.2))

#> Derivative at current rate of 70%
print(f"f'(0.7) = {revenue_derivative(0.7):.2f}")
```

```
f'(0.7) = -41.18
```

$f'(0.7) < 0$ → revenue is **decreasing** at 70%. We should **lower** the tax rate!

What Does the Derivative Sign Tell Us?

Negative derivative at the red point:

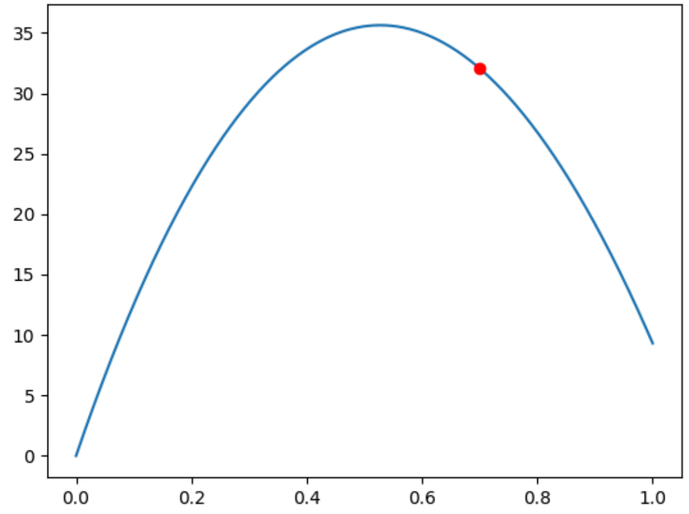
→ Function is going **downhill**

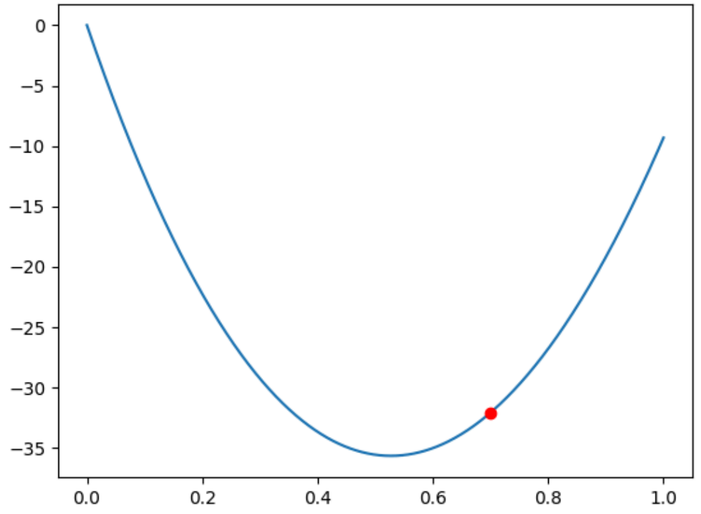
→ To increase f, move **left** (decrease x)

Positive derivative at the red point:

→ Function is going **uphill**

→ To increase f, move **right** (increase x)





Taking a Step in the Right Direction

Since $f'(0.7) < 0$, we should **decrease** the tax rate.

Key idea: Move in the direction **opposite** to the negative derivative.

```
current_rate = 0.7
step_size = 0.003

#> Take one step: move opposite to negative derivative
current_rate = current_rate + step_size * revenue_derivative(current_rate)
print(f"New rate: {current_rate:.4f}")
print(f"New revenue: {revenue(current_rate):.2f}")
```

```
New rate: 0.5765
New revenue: 35.35
```

Revenue **increased**! Can we keep going?

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The Gradient Ascent Update Rule

Repeat the step-taking process until convergence:

$$x_{\text{new}} = x_{\text{current}} + \alpha \cdot \nabla f(x_{\text{current}})$$

Where:

- α = **step size** (learning rate) — controls how big each step is
- $\nabla f(x)$ = **gradient** (derivative) — points in the direction of steepest **ascent**
- If $\nabla f > 0$: move **right** (increase x)
- If $\nabla f < 0$: move **left** (decrease x)

Intuition: Always walk uphill. Stop when the ground is flat ($\nabla f \approx 0$).

Gradient Ascent Algorithm

Step	Action
1	Choose initial x_0 and step size α
2	Compute gradient $\nabla f(x_k)$
3	Update: $x_{k+1} = x_k + \alpha \cdot \nabla f(x_k)$
4	If $\lvert\alpha \cdot \nabla f\rvert < \text{threshold}$ → **stop**
5	Otherwise → go to Step 2

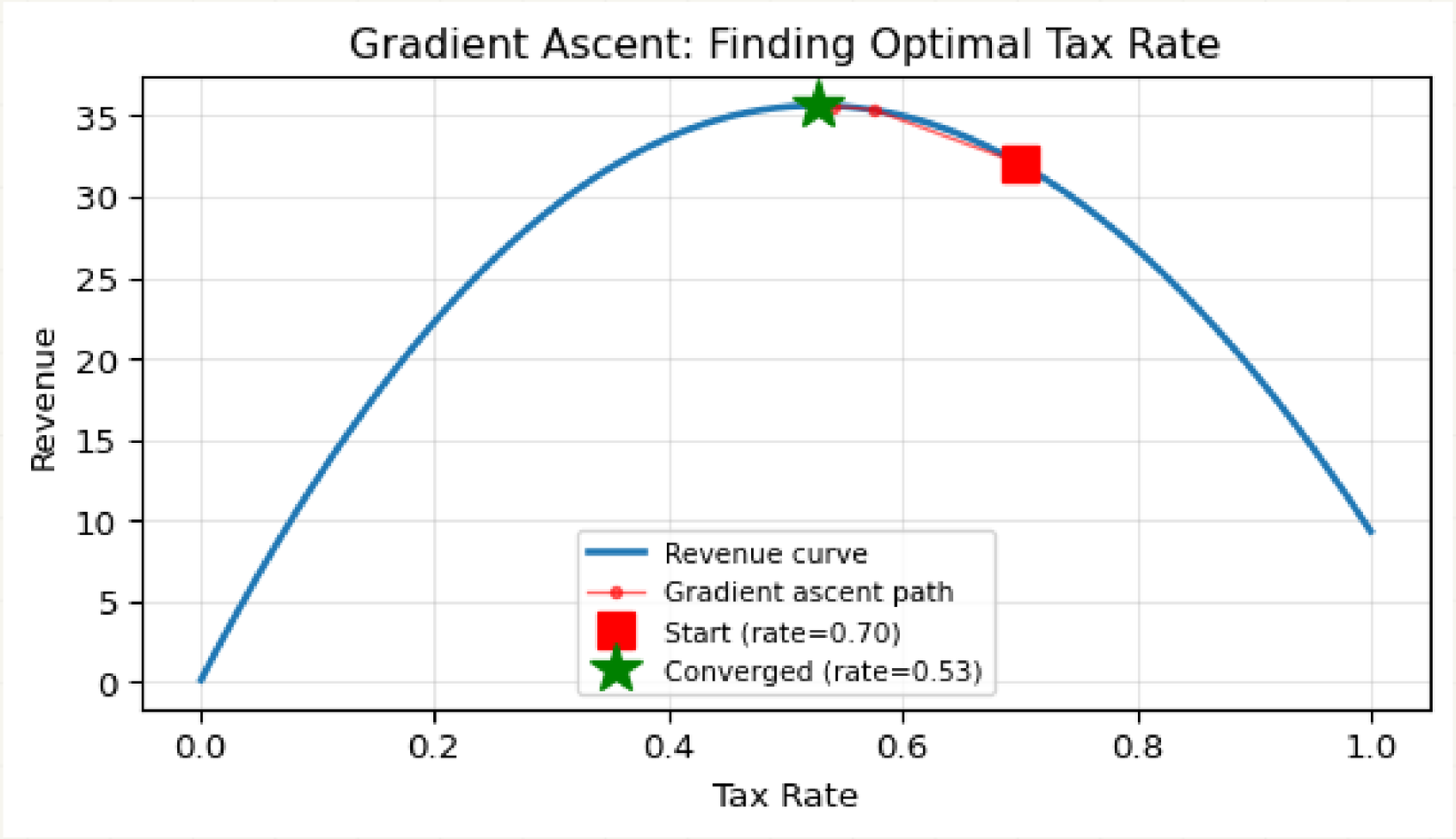
Stopping rule: When the change is smaller than a threshold, we've converged.

Gradient Ascent in Python

```python
current_rate = 0.7
step_size = 0.003
threshold = 0.0001

for i in range(1000):
    rate_change = step_size * revenue_derivative(current_rate)
    current_rate = current_rate + rate_change

    if abs(rate_change) < threshold:
        print(f"Converged after {i+1} iterations")
        break

print(f"Optimal tax rate: {current_rate:.4f}")
print(f"Maximum revenue:  {revenue(current_rate):.2f}")
```

```
Converged after 7 iterations
Optimal tax rate: 0.5274
Maximum revenue:  35.64
```

Visualizing Gradient Ascent



Starting from 70%, gradient ascent converges to the **optimal tax rate** in a few dozen steps.

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Local vs. Global Maximum

Gradient ascent finds a **local** maximum — the nearest peak.

But that may not be the **global** maximum!

The Problem:

- **Local Peak**: Higher than immediate surroundings, but not the highest overall
- **Mountain Analogy**: Climbing to a foothill summit — you'd need to go **downhill** first to find a taller peak
- **Gradient ascent never goes downhill** → it can get stuck

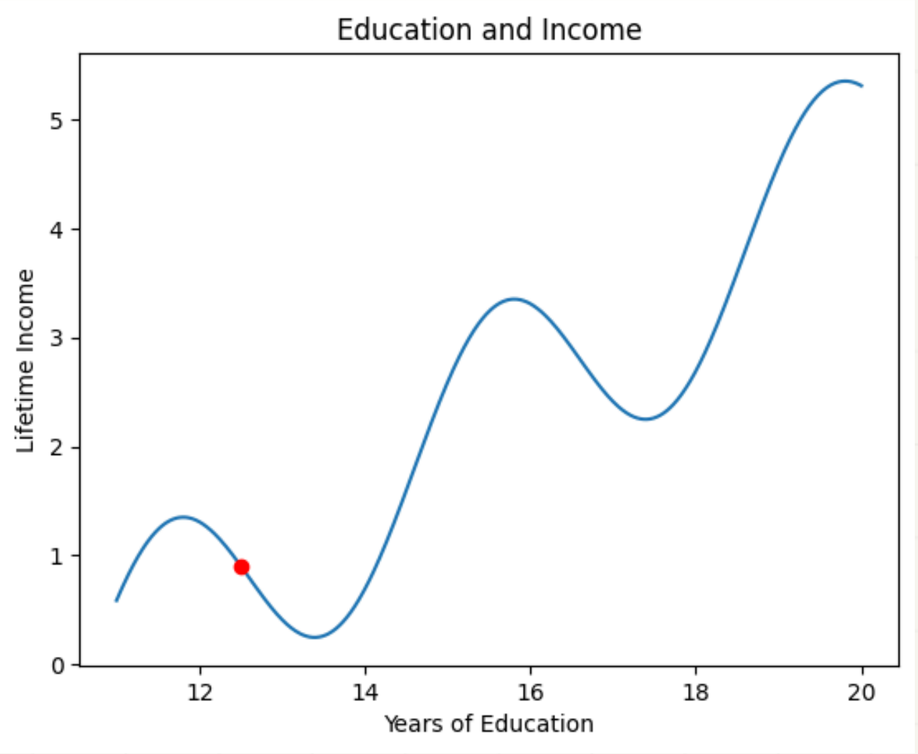
No simple fix! This is a **fundamental limitation** of greedy local search.

Education and Income: A Real-World Analogy

The Trap of Local Optima:

- Dropping out of school gives **immediate** income
- But staying in school → **higher** long-term earnings
- You must **temporarily decrease** income to reach the global maximum

Gradient ascent is **greedy** — it can't see the bigger picture.



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Gradient Descent: Flipping the Sign

In machine learning, we usually want to **minimize** a loss function, not maximize.

Two ways to switch:

Option 1: Negate the function then maximize negative of $f(x)$ instead

Option 2: Flip the update rule sign:

$$x_{\text{new}} = x_{\text{current}} - \alpha \cdot \nabla f(x_{\text{current}})$$

That's it! One sign change turns gradient **ascent** into gradient **descent**.

This is the foundation of training neural networks.

Gradient Descent in Python

```python
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return np.sin(5*x)*x + np.cos(2*x)*x + np.sin(15*x)

def numerical_derivative(func, a, h=0.0001):
    return (func(a + h) - func(a - h)) / (2 * h)

#> Gradient Descent
x_current = 1.0
alpha = 0.001
for i in range(1000):
    x_current = x_current - alpha * numerical_derivative(f, x_current)

print(f"Local minimum at x = {x_current:.4f}")
print(f"f(x) = {f(x_current):.4f}")
```

```
Local minimum at x = 1.1434
f(x) = -2.3557
```

Same algorithm, opposite direction. **Gradient descent** is the workhorse of modern AI.

Summary

Concept	Key Takeaway
Derivative	Measures slope — tells us which direction is “uphill”
Gradient Ascent	$x_{\text{new}} = x + \alpha \nabla f(x)$ — walk uphill to maximize
Local Extrema	Greedy search can get stuck — no guarantee of global optimum
Gradient Descent	$x_{\text{new}} = x - \alpha \nabla f(x)$ — walk downhill to minimize

Next up: Extending these ideas to **multivariable functions** and **partial derivatives**.

Thank You!